5.8 Hyperbolic Functions

- Develop properties of hyperbolic functions.
- Differentiate and integrate hyperbolic functions.
- Develop properties of inverse hyperbolic functions.
- Differentiate and integrate functions involving inverse hyperbolic functions.

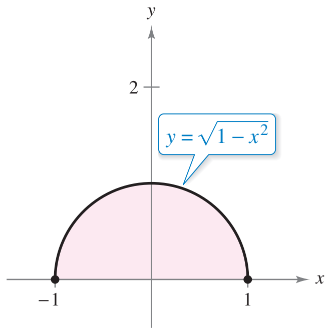
JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

See LarsonCalculus.com to read more of this biography.

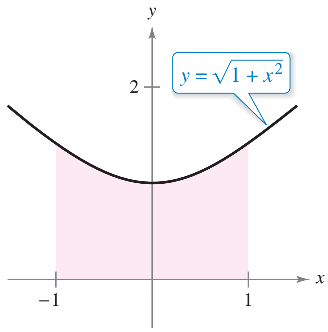
Hyperbolic Functions

In this section, you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.29, with the area of a region under a hyperbola, as shown in Figure 5.30.



Circle: $x^2 + y^2 = 1$

Figure 5.29



Hyperbola: $-x^2 + y^2 = 1$

Figure 5.30

The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{1 - x^2} + \arcsin x\right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1 + x^2}\, dx = \frac{1}{2}\left[x\sqrt{1 + x^2} + \sinh^{-1} x\right]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.

REMARK The notation $\sinh x$ is read as "the hyperbolic sine of x," $\cosh x$ as "the hyperbolic cosine of x," and so on.

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

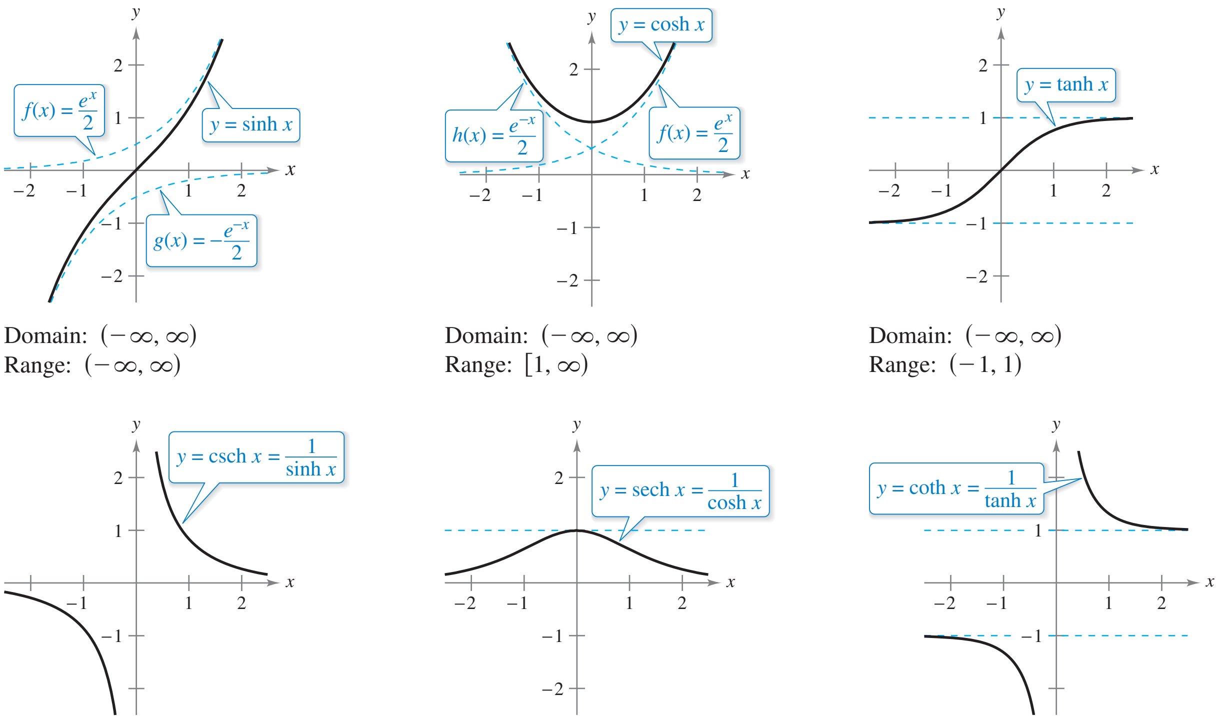
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article "An Introduction to Hyperbolic Functions in Elementary Calculus" by Jerome Rosenthal in *Mathematics Teacher.* To view this article, go to *MathArticles.com.*

American Institute of Physics (AIP) (use Emilio Serge Visual Archive)

The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.31. Note that the graph of $\sinh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph of $\cosh x$ can be obtained by adding the corresponding y-coordinates of the exponential functions $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Domain: $(-\infty, \infty)$
Range: $[1, \infty)$

Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Domain: $(-\infty, \infty)$
Range: $(0, 1]$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Figure 5.31

Many of the trigonometric identities have corresponding *hyperbolic identities.* For instance,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} \\ &= 1.\end{aligned}$$

FOR FURTHER INFORMATION To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review.*

HYPERBOLIC IDENTITIES

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\tanh^2 x + \operatorname{sech}^2 x = 1 \qquad \sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$
$$\coth^2 x - \operatorname{csch}^2 x = 1 \qquad \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$
$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$
$$\sinh^2 x = \frac{-1 + \cosh 2x}{2} \qquad \cosh^2 x = \frac{1 + \cosh 2x}{2}$$
$$\sinh 2x = 2 \sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The next theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.18 Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Proof Here is a proof of two of the differentiation rules. (You are asked to prove some of the other differentiation rules in Exercises 103–105.)

$$\begin{aligned}\frac{d}{dx}[\sinh x] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right]\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[\tanh x] &= \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right]\\ &= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Differentiation of Hyperbolic Functions

a. $\frac{d}{dx}[\sinh(x^2 - 3)] = 2x \cosh(x^2 - 3)$

b. $\frac{d}{dx}[\ln(\cosh x)] = \frac{\sinh x}{\cosh x} = \tanh x$

c. $\frac{d}{dx}[x \sinh x - \cosh x] = x \cosh x + \sinh x - \sinh x = x \cosh x$

d. $\frac{d}{dx}[(x - 1)\cosh x - \sin x] = (x - 1)\sinh x + \cosh x - \cosh x = (x - 1)\sinh x$

EXAMPLE 2 Finding Relative Extrema

Find the relative extrema of

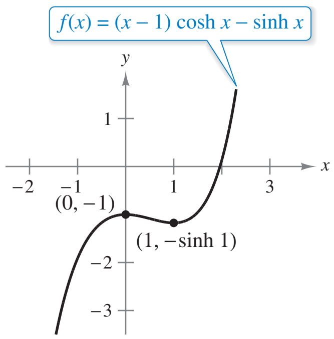
$$f(x) = (x - 1)\cosh x - \sinh x.$$

Solution Using the result of Example 1(d), set the first derivative of f equal to 0.

$$(x - 1)\sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, you can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh 1)$ yields a relative minimum, as shown in Figure 5.32. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions, as shown.

$$\begin{aligned} f(x) &= (x - 1)\left(\frac{1}{2}\right)(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) \\ &= \frac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \frac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.32

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a *catenary*, as discussed in Example 3.

EXAMPLE 3 Hanging Power Cables

See LarsonCalculus.com for an interactive version of this type of example.

Power cables are suspended between two towers, forming the catenary shown in Figure 5.33. The equation for this catenary is

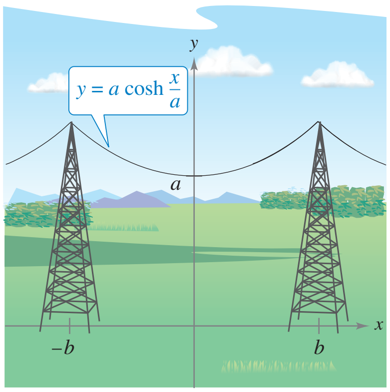
$$y = a\cosh\frac{x}{a}.$$

The distance between the two towers is $2b$. Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right)\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

At the point $(b, a\cosh(b/a))$, the slope (from the left) is $m = \sinh\dfrac{b}{a}$.



Catenary
Figure 5.33

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article "Reexamining the Catenary" by Paul Cella in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 4 Integrating a Hyperbolic Function

Find $\displaystyle\int \cosh 2x \sinh^2 2x\, dx$.

Solution

$$\begin{aligned} \int \cosh 2x \sinh^2 2x\, dx &= \frac{1}{2}\int (\sinh 2x)^2(2\cosh 2x)\, dx \qquad u = \sinh 2x \\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C \\ &= \frac{\sinh^3 2x}{6} + C \end{aligned}$$

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.31, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one when their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.19.

THEOREM 5.19 Inverse Hyperbolic Functions

Function	Domain
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$	$(-1, 1)$
$\coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x + 1}{x - 1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln \dfrac{1 + \sqrt{1 - x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

Proof The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, for

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

and

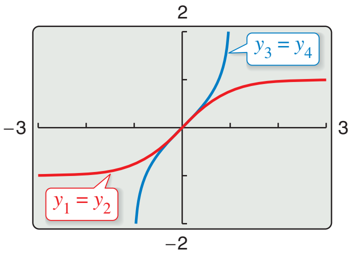
$$g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

you can show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

which implies that g is the inverse function of f.

See LarsonCalculus.com for Bruce Edwards's video of this proof.



Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function
Figure 5.34

TECHNOLOGY You can use a graphing utility to confirm graphically the results of Theorem 5.19. For instance, graph the following functions.

$y_1 = \tanh x$ — Hyperbolic tangent

$y_2 = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ — Definition of hyperbolic tangent

$y_3 = \tanh^{-1} x$ — Inverse hyperbolic tangent

$y_4 = \dfrac{1}{2} \ln \dfrac{1 + x}{1 - x}$ — Definition of inverse hyperbolic tangent

The resulting display is shown in Figure 5.34. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

The graphs of the inverse hyperbolic functions are shown in Figure 5.35.

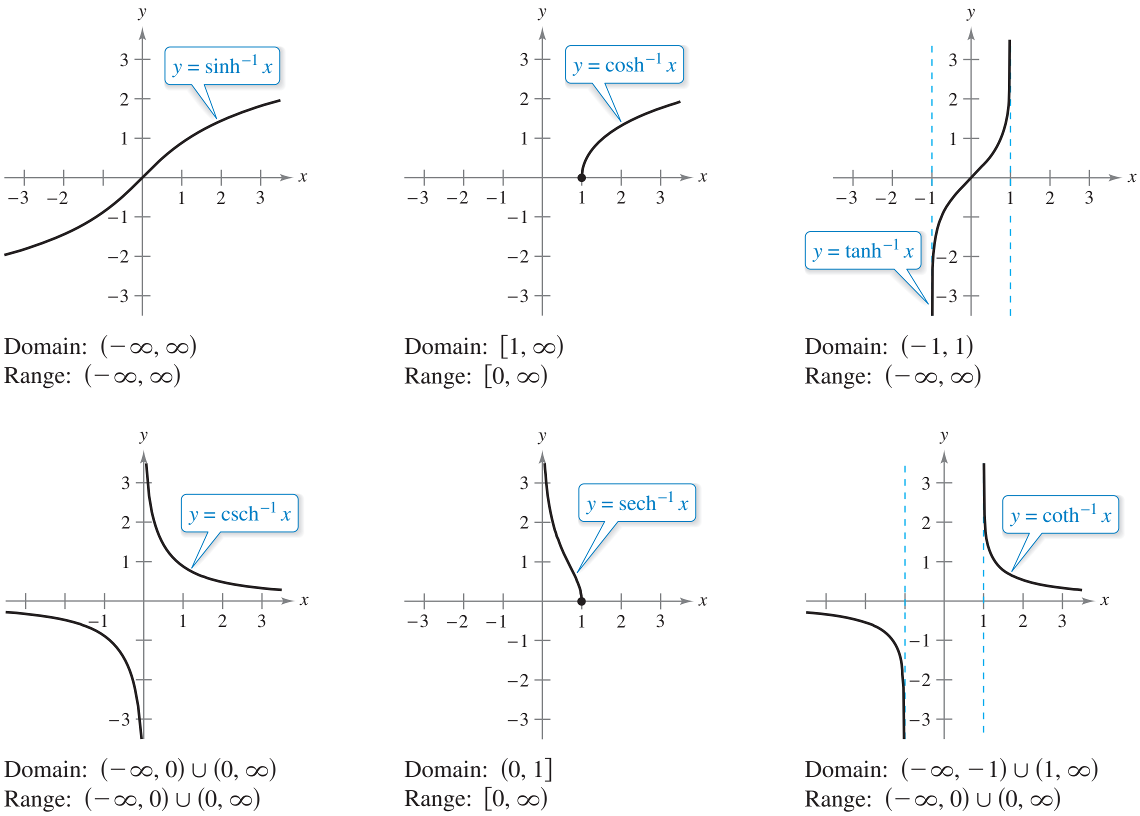


Figure 5.35

The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.

EXAMPLE 5 A Tractrix

A person is holding a rope that is tied to a boat, as shown in Figure 5.36. As the person walks along the dock, the boat travels along a **tractrix,** given by the equation

$$y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$$

where a is the length of the rope. For $a = 20$ feet, find the distance the person must walk to bring the boat to a position 5 feet from the dock.

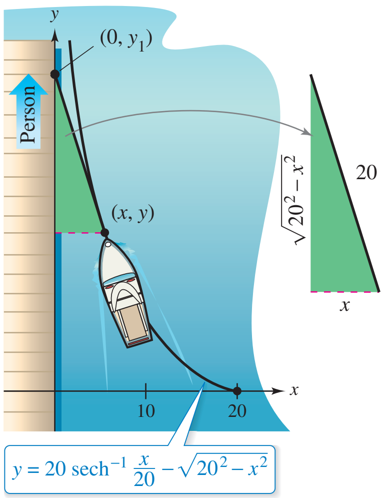
Solution In Figure 5.36, notice that the distance the person has walked is

$$\begin{aligned} y_1 &= y + \sqrt{20^2 - x^2} \\ &= \left(20 \operatorname{sech}^{-1} \frac{x}{20} - \sqrt{20^2 - x^2}\right) + \sqrt{20^2 - x^2} \\ &= 20 \operatorname{sech}^{-1} \frac{x}{20}. \end{aligned}$$

When $x = 5$, this distance is

$$y_1 = 20 \operatorname{sech}^{-1} \frac{5}{20} = 20 \ln \frac{1 + \sqrt{1 - (1/4)^2}}{1/4} = 20 \ln\left(4 + \sqrt{15}\right) \approx 41.27 \text{ feet.}$$

So, the person must walk about 41.27 feet to bring the boat to a position 5 feet from the dock.



A person must walk about 41.27 feet to bring the boat to a position 5 feet from the dock.
Figure 5.36

Inverse Hyperbolic Functions: Differentiation and Integration

The derivatives of the inverse hyperbolic functions, which resemble the derivatives of the inverse trigonometric functions, are listed in Theorem 5.20 with the corresponding integration formulas (in logarithmic form). You can verify each of these formulas by applying the logarithmic definitions of the inverse hyperbolic functions. (See Exercises 106–108.)

THEOREM 5.20 Differentiation and Integration Involving Inverse Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1-u^2} \qquad \frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1-u^2}} \qquad \frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

EXAMPLE 6 Differentiation of Inverse Hyperbolic Functions

a. $$\frac{d}{dx}[\sinh^{-1}(2x)] = \frac{2}{\sqrt{(2x)^2+1}} = \frac{2}{\sqrt{4x^2+1}}$$

b. $$\frac{d}{dx}[\tanh^{-1}(x^3)] = \frac{3x^2}{1-(x^3)^2} = \frac{3x^2}{1-x^6}$$

EXAMPLE 7 Integration Using Inverse Hyperbolic Functions

a. $$\int \frac{dx}{x\sqrt{4-9x^2}} = \int \frac{3\,dx}{(3x)\sqrt{4-9x^2}} \qquad \int \frac{du}{u\sqrt{a^2-u^2}}$$

$$= -\frac{1}{2}\ln\frac{2+\sqrt{4-9x^2}}{|3x|} + C \qquad -\frac{1}{a}\ln\frac{a+\sqrt{a^2-u^2}}{|u|} + C$$

REMARK Let $a = 2$ and $u = 3x$.

b. $$\int \frac{dx}{5-4x^2} = \frac{1}{2}\int \frac{2\,dx}{(\sqrt{5})^2 - (2x)^2} \qquad \int \frac{du}{a^2-u^2}$$

$$= \frac{1}{2}\left(\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x}{\sqrt{5}-2x}\right|\right) + C \qquad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$= \frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x}{\sqrt{5}-2x}\right| + C$$

REMARK Let $a = \sqrt{5}$ and $u = 2x$.

5.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Function **In Exercises 1–6, evaluate the function. If the value is not a rational number, round your answer to three decimal places.**

1. (a) $\sinh 3$ (b) $\tanh(-2)$
2. (a) $\cosh 0$ (b) $\operatorname{sech} 1$
3. (a) $\operatorname{csch}(\ln 2)$ (b) $\coth(\ln 5)$
4. (a) $\sinh^{-1} 0$ (b) $\tanh^{-1} 0$
5. (a) $\cosh^{-1} 2$ (b) $\operatorname{sech}^{-1} \frac{2}{3}$
6. (a) $\operatorname{csch}^{-1} 2$ (b) $\coth^{-1} 3$

Verifying an Identity **In Exercises 7–14, verify the identity.**

7. $\tanh^2 x + \operatorname{sech}^2 x = 1$
8. $\coth^2 x - \operatorname{csch}^2 x = 1$
9. $\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$
10. $\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$
11. $\sinh 2x = 2 \sinh x \cosh x$
12. $e^{2x} = \sinh 2x + \cosh 2x$
13. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
14. $\cosh x + \cosh y = 2 \cosh \dfrac{x + y}{2} \cosh \dfrac{x - y}{2}$

Finding Values of Hyperbolic Functions **In Exercises 15 and 16, use the value of the given hyperbolic function to find the values of the other hyperbolic functions at x.**

15. $\sinh x = \dfrac{3}{2}$
16. $\tanh x = \dfrac{1}{2}$

Finding a Limit **In Exercises 17–22, find the limit.**

17. $\lim\limits_{x\to\infty} \sinh x$
18. $\lim\limits_{x\to-\infty} \tanh x$
19. $\lim\limits_{x\to\infty} \operatorname{sech} x$
20. $\lim\limits_{x\to-\infty} \operatorname{csch} x$
21. $\lim\limits_{x\to 0} \dfrac{\sinh x}{x}$
22. $\lim\limits_{x\to 0^-} \coth x$

Finding a Derivative **In Exercises 23–32, find the derivative of the function.**

23. $f(x) = \sinh 3x$
24. $f(x) = \cosh(8x + 1)$
25. $y = \operatorname{sech}(5x^2)$
26. $f(x) = \tanh(4x^2 + 3x)$
27. $f(x) = \ln(\sinh x)$
28. $y = \ln\left(\tanh \dfrac{x}{2}\right)$
29. $h(x) = \dfrac{1}{4} \sinh 2x - \dfrac{x}{2}$
30. $y = x \cosh x - \sinh x$
31. $f(t) = \arctan(\sinh t)$
32. $g(x) = \operatorname{sech}^2 3x$

Finding an Equation of a Tangent Line **In Exercises 33–36, find an equation of the tangent line to the graph of the function at the given point.**

33. $y = \sinh(1 - x^2), \quad (1, 0)$
34. $y = x^{\cosh x}, \quad (1, 1)$
35. $y = (\cosh x - \sinh x)^2, \quad (0, 1)$
36. $y = e^{\sinh x}, \quad (0, 1)$

Finding Relative Extrema **In Exercises 37–40, find any relative extrema of the function. Use a graphing utility to confirm your result.**

37. $f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$
38. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$
39. $g(x) = x \operatorname{sech} x$
40. $h(x) = 2 \tanh x - x$

Catenary **In Exercises 41 and 42, a model for a power cable suspended between two towers is given. (a) Graph the model, (b) find the heights of the cable at the towers and at the midpoint between the towers, and (c) find the slope of the model at the point where the cable meets the right-hand tower.**

41. $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$
42. $y = 18 + 25 \cosh \dfrac{x}{25}, \quad -25 \le x \le 25$

Finding an Indefinite Integral **In Exercises 43–54, find the indefinite integral.**

43. $\displaystyle\int \cosh 2x \, dx$
44. $\displaystyle\int \operatorname{sech}^2(3x) \, dx$
45. $\displaystyle\int \sinh(1 - 2x) \, dx$
46. $\displaystyle\int \frac{\cosh \sqrt{x}}{\sqrt{x}} \, dx$
47. $\displaystyle\int \cosh^2(x - 1) \sinh(x - 1) \, dx$
48. $\displaystyle\int \frac{\sinh x}{1 + \sinh^2 x} \, dx$
49. $\displaystyle\int \frac{\cosh x}{\sinh x} \, dx$
50. $\displaystyle\int \operatorname{sech}^2(2x - 1) \, dx$
51. $\displaystyle\int x \operatorname{csch}^2 \frac{x^2}{2} \, dx$
52. $\displaystyle\int \operatorname{sech}^3 x \tanh x \, dx$
53. $\displaystyle\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2} \, dx$
54. $\displaystyle\int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} \, dx$

Evaluating a Definite Integral **In Exercises 55–60, evaluate the integral.**

55. $\displaystyle\int_0^{\ln 2} \tanh x \, dx$
56. $\displaystyle\int_0^1 \cosh^2 x \, dx$
57. $\displaystyle\int_0^4 \frac{1}{25 - x^2} \, dx$
58. $\displaystyle\int_0^4 \frac{1}{\sqrt{25 - x^2}} \, dx$
59. $\displaystyle\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}} \, dx$
60. $\displaystyle\int_0^{\ln 2} 2e^{-x} \cosh x \, dx$

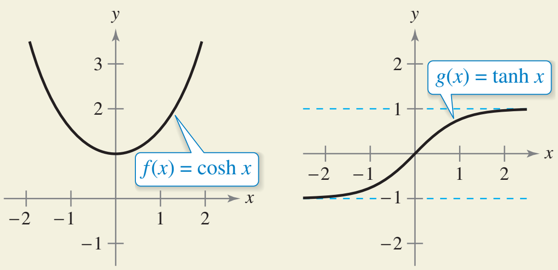
WRITING ABOUT CONCEPTS

61. Comparing Functions Discuss several ways in which the hyperbolic functions are similar to the trigonometric functions.

62. Hyperbolic Functions Which hyperbolic functions take on only positive values? Which hyperbolic functions are increasing on their domains?

63. Comparing Derivative Formulas Which hyperbolic derivative formulas differ from their trigonometric counterparts by a minus sign?

64. HOW DO YOU SEE IT? Use the graphs of f and g shown in the figures to answer the following.



(a) Identify the open interval(s) on which the graphs of f and g are increasing or decreasing.

(b) Identify the open interval(s) on which the graphs of f and g are concave upward or concave downward.

Finding a Derivative **In Exercises 65–74, find the derivative of the function.**

65. $y = \cosh^{-1}(3x)$

66. $y = \tanh^{-1} \dfrac{x}{2}$

67. $y = \tanh^{-1} \sqrt{x}$

68. $f(x) = \coth^{-1}(x^2)$

69. $y = \sinh^{-1}(\tan x)$

70. $y = \tanh^{-1}(\sin 2x)$

71. $y = (\operatorname{csch}^{-1} x)^2$

72. $y = \operatorname{sech}^{-1}(\cos 2x), \quad 0 < x < \pi/4$

73. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

74. $y = x \tanh^{-1} x + \ln \sqrt{1 - x^2}$

Finding an Indefinite Integral **In Exercises 75–82, find the indefinite integral using the formulas from Theorem 5.20.**

75. $\displaystyle\int \frac{1}{3 - 9x^2}\,dx$

76. $\displaystyle\int \frac{1}{2x\sqrt{1 - 4x^2}}\,dx$

77. $\displaystyle\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx$

78. $\displaystyle\int \frac{x}{9 - x^4}\,dx$

79. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx$

80. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx$

81. $\displaystyle\int \frac{-1}{4x - x^2}\,dx$

82. $\displaystyle\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 8}}$

Evaluating a Definite Integral **In Exercises 83–86, evaluate the definite integral using the formulas from Theorem 5.20.**

83. $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2 - 4}}\,dx$

84. $\displaystyle\int_1^3 \frac{1}{x\sqrt{4 + x^2}}\,dx$

85. $\displaystyle\int_{-1}^1 \frac{1}{16 - 9x^2}\,dx$

86. $\displaystyle\int_0^1 \frac{1}{\sqrt{25x^2 + 1}}\,dx$

Differential Equation **In Exercises 87–90, solve the differential equation.**

87. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{80 + 8x - 16x^2}}$

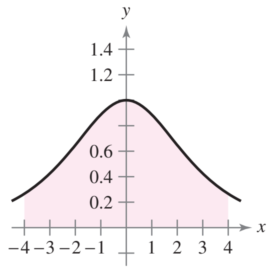
88. $\dfrac{dy}{dx} = \dfrac{1}{(x - 1)\sqrt{-4x^2 + 8x - 1}}$

89. $\dfrac{dy}{dx} = \dfrac{x^3 - 21x}{5 + 4x - x^2}$

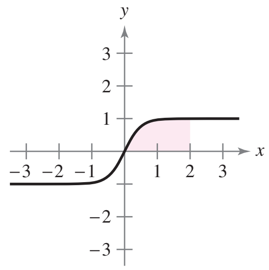
90. $\dfrac{dy}{dx} = \dfrac{1 - 2x}{4x - x^2}$

Area **In Exercises 91–94, find the area of the region.**

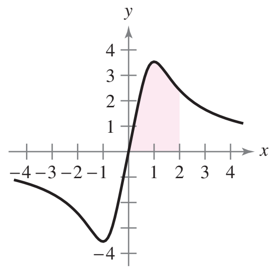
91. $y = \operatorname{sech} \dfrac{x}{2}$



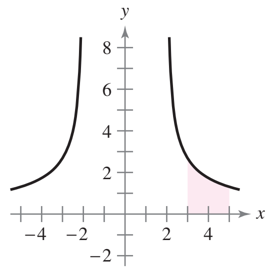
92. $y = \tanh 2x$



93. $y = \dfrac{5x}{\sqrt{x^4 + 1}}$



94. $y = \dfrac{6}{\sqrt{x^2 - 4}}$



95. Chemical Reactions Chemicals A and B combine in a 3-to-1 ratio to form a compound. The amount of compound x being produced at any time t is proportional to the unchanged amounts of A and B remaining in the solution. So, when 3 kilograms of A is mixed with 2 kilograms of B, you have

$$\frac{dx}{dt} = k\left(3 - \frac{3x}{4}\right)\left(2 - \frac{x}{4}\right) = \frac{3k}{16}(x^2 - 12x + 32).$$

One kilogram of the compound is formed after 10 minutes. Find the amount formed after 20 minutes by solving the equation

$$\int \frac{3k}{16}\,dt = \int \frac{dx}{x^2 - 12x + 32}.$$

96. Vertical Motion An object is dropped from a height of 400 feet.

(a) Find the velocity of the object as a function of time (neglect air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of the velocity, then $dv/dt = -32 + kv^2$, where -32 feet per second per second is the acceleration due to gravity and k is a constant. Show that the velocity v as a function of time is $v(t) = -\sqrt{32/k}\tanh\left(\sqrt{32k}\,t\right)$ by performing $\int dv/(32 - kv^2) = -\int dt$ and simplifying the result.

(d) Use the result of part (c) to find $\lim_{t\to\infty} v(t)$ and give its interpretation.

(e) Integrate the velocity function in part (c) and find the position s of the object as a function of t. Use a graphing utility to graph the position function when $k = 0.01$ and the position function in part (b) in the same viewing window. Estimate the additional time required for the object to reach ground level when air resistance is not neglected.

(f) Give a written description of what you believe would happen if k were increased. Then test your assertion with a particular value of k.

97. Tractrix **Consider the equation of the tractrix**

$$y = a\,\mathrm{sech}^{-1}(x/a) - \sqrt{a^2 - x^2}, \quad a > 0.$$

(a) Find dy/dx.

(b) Let L be the tangent line to the tractrix at the point P. When L intersects the y-axis at the point Q, show that the distance between P and Q is a.

98. Tractrix Show that the boat in Example 5 is always pointing toward the person.

99. Proof Prove that

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad -1 < x < 1.$$

100. Proof Prove that

$$\sinh^{-1} t = \ln\left(t + \sqrt{t^2 + 1}\right).$$

101. Using a Right Triangle Show that

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

102. Integration Let $x > 0$ and $b > 0$. Show that

$$\int_{-b}^{b} e^{xt}\,dt = \frac{2\sinh bx}{x}.$$

Proof **In Exercises 103–105, prove the differentiation formula.**

103. $\frac{d}{dx}[\cosh x] = \sinh x$

104. $\frac{d}{dx}[\coth x] = -\mathrm{csch}^2\, x$

105. $\frac{d}{dx}[\mathrm{sech}\, x] = -\mathrm{sech}\, x \tanh x$

Verifying a Differentiation Rule **In Exercises 106–108, verify the differentiation formula.**

106. $\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2 - 1}}$

107. $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2 + 1}}$

108. $\frac{d}{dx}[\mathrm{sech}^{-1} x] = \frac{-1}{x\sqrt{1 - x^2}}$

PUTNAM EXAM CHALLENGE

109. From the vertex $(0, c)$ of the catenary $y = c\cosh(x/c)$ a line L is drawn perpendicular to the tangent to the catenary at point P. Prove that the length of L intercepted by the axes is equal to the ordinate y of the point P.

110. Prove or disprove: there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

SECTION PROJECT

St. Louis Arch

The Gateway Arch in St. Louis, Missouri, was constructed using the hyperbolic cosine function. The equation used for construction was

$$y = 693.8597 - 68.7672 \cosh 0.0100333x,$$
$$-299.2239 \le x \le 299.2239$$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is

$$A = 125.1406 \cosh 0.0100333x.$$

(Source: Owner's Manual for the Gateway Arch, Saint Louis, MO, by William Thayer)

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?

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